DIFFUSIVE FLOW TO A SPHERE AT SMALL AND MODERATE REYNOLDS NUMBERS

APPROXIMATION OF A DIFFUSION BOUNDARY LAYER

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The relative importance of purely diffusive flow to a spherical object is often of interest in connection with several problems in the physics and physical chemistry of the atmosphere [1, 2]. Molecular transport in very viscous liquids and the transport of small aerosol particles ($r_1 < 10^{-4}$ cm) in gases by means of Browian motion are described by the same equations (when linkage of particle is neglected). We note that the deposition of aerosol particles on a spherical object of radius r_2, due to linkage, is of the order $(r_1/r_2)^2$, so this effect may be neglected if the purely diffusive flow is at least an order of magnitude greater than $(r_1/r_2)^2$.

The similarity criteria for this problem are the Reynolds number R of the flow and the Peclet number λ:

$$R = r_2 u_\infty / \nu, \quad \lambda = r_2 u_\infty / D$$

Here u_∞ is the velocity of the unperturbed flow, ν is the kinematic viscosity of the medium, and D is the diffusion coefficient of the substance. For the case $R \ll 1$, in which the Stokes approximation may be used for the hydrodynamic velocity field u_∞ in steady-state flow around a sphere, and $\lambda \gg 1$ (in the approximation of a diffusion boundary layer), the following equation was obtained in [1] for the dimensionless integral flow:

$$I \approx 7.848\lambda^{-2/3}$$

Below we will refine this equation for the case of moderate R ($R \le 20$). At such R, the flow is of a more or less smooth nature: an eddy begins to form behind the sphere at $R \approx 8$; $R \approx 20$, it still makes up a small part of the flow. We note that for a drop of water falling freely in air, we have $R \approx 20$ at $r_2 \approx 0.02$ cm.

In the spherical coordinate system $\{\xi, \theta, \varphi\}$ (ξ is the distance from the sphere, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$), the basic equations for the steady-state problem are

$$\lambda u_\xi \frac{\partial n}{\partial \xi} + \lambda \frac{u_\theta}{1+\xi} \frac{\partial n}{\partial \theta} = \frac{\partial^2 n}{\partial \xi^2} + \frac{2}{1+\xi} \frac{\partial n}{\partial \xi} + \frac{1}{(1+\xi)^2} \left[\frac{\partial^2}{\partial \theta^2} + \operatorname{ctg} \theta \frac{\partial}{\partial \theta} \right] n \quad (1)$$

$$I_1 = u_\infty n_\infty r_2^2 I, \qquad I = \frac{2\pi}{\lambda} \int_0^\pi j(\theta) \sin \theta \, d\theta, \qquad j = \left. \frac{\partial n}{\partial \xi} \right|_{\xi=0} \quad (2)$$

$$n \to 1, \quad \xi \to \infty; \qquad n \to 0, \quad \xi \to 0 \quad (3)$$

$$u_\xi \to 1, \quad u_\theta \to 0, \quad \xi \to \infty$$

$$u_\xi \to u_1 \xi^2, \qquad u_\theta \to u_2 \xi, \qquad \xi \to 0 \quad (4)$$

$$u_1 + \frac{1}{2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta u_2 = 0$$

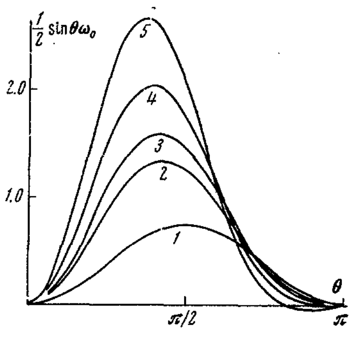

Fig. 1

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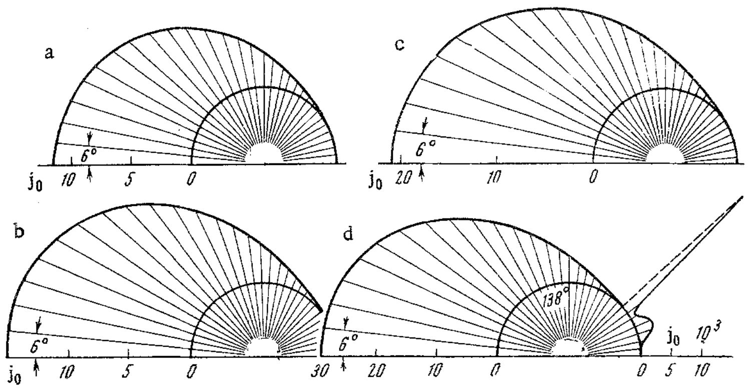

Fig. 2

We note that Eqs. (4) are not limited: they always hold for viscous flow around a sufficiently smooth bounded surface. Using these equations and converting Eqs. (1) to those of a diffusion boundary layer by the familiar method, we find

$$u_1\zeta^2\frac{\partial n_0}{\partial\zeta}+u_2\zeta\frac{\partial n_0}{\partial\theta}=\frac{\partial^2 n_0}{\partial\zeta^2}+O(\lambda^{-1/3}) \tag{5}$$

$$I=2\pi\lambda^{-2/3}\int_0^\pi j_0(\theta)\sin\theta\,d\theta+O(\lambda^{-1})$$

$$j_0=\frac{\partial n_0}{\partial\zeta}\bigg|_{\zeta=0},\qquad \zeta=\xi\lambda^{1/3}$$

$$n_0\to 1,\quad \zeta\to\infty,\quad \theta<\pi$$

$$n_0\to 0,\quad \zeta\to 0;\qquad n\to 0,\quad 0\leqslant\zeta<\infty,\quad \theta\to\pi$$

The solution of Eq. (5) is

$$n_0=1-\frac{\Gamma(^1/_3,\ \zeta^3/\mu)}{\Gamma(^1/_3)} \tag{6}$$

$$\mu=9u_2^{-3/2}\sin^{-3/2}\theta\int_0^\theta \sin^{3/2}\theta\,u_2^{1/2}\,d\theta$$

$$j_0=\frac{1}{\Gamma(1+{}^1/_3)}\mu^{-1/3},\qquad I=\frac{2\pi}{\Gamma(1+{}^1/_3)}\lambda^{-2/3}\left[\int_0^\pi\frac{\sin\theta}{\mu^{1/3}}\,d\theta+O(\lambda^{-1/3})\right]$$

$$\left(\Gamma(^1/_3,\ x)=\int_x^\infty z^{-2/3}e^{-z}\,dz\right)$$

For R $\leqslant$ 8, we have $u_2 > 0$ for $0 \le \theta \le \pi$; for this case, we thus have

$$I=\frac{3^{1/3}\pi\ \lambda^{-2/3}}{\Gamma(1+{}^1/_3)}\left[\int_0^\pi \sin^{3/2}\theta\,u_2^{1/2}\,d\theta\right]^{2/3}+O(\lambda^{-1}) \tag{7}$$

For R $\geqslant$ 8, u_2 becomes negative in a certain θ range: $u_2 < 0$, $\theta_0 < \theta \le \pi$. We will show below that the contribution to I from the internal $\theta_0 < \theta \le \pi$ is negligibly small, so we can replace the upper limit in Eq. (7) by θ_0 in calculating the integral flow. It should also be noted that as $\theta \to \pi$, Eq. (6) becomes inapplicable. The important change in n occurs at distances of the order of $\mu\lambda^{-1/3}$ from the sphere, while μ increases without bound as $\theta \to \pi$. Since the first terms in the expansion of u_ξ and u_θ in powers of ξ were used in deriving Eq. (6), it cannot correctely reflect the behavior of n at the bottom of the flow. In addition, the inequality

$$\partial n_0/\partial\theta \neq 0,\qquad \theta\to\pi \tag{8}$$

which contradicts the axial-symmetry conditions, follows from (6).

Inequality (8) yields an important result. Although the behavior of n at the "bottom" of the flow is not described correctly by Eq. (6), there is no reason to doubt the correctness of asymptotic relation (7) for the

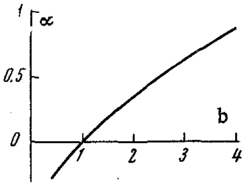

Fig. 3

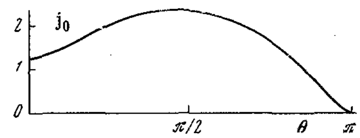

Fig. 4

integral flow. Moreover, using a procedure analogous to that in [3] in an examination of diffusion from a viscous flow toward a circular cylinder (and toward a system of cylinders), we can in principle construct for I the asymptotic expansion

$$I \approx \sum_{(0 \leqslant m \leqslant N)} I^{(m)} \lambda^{-(m+2)/3} \tag{9}$$

$$I^{(m)} = 2\pi \int_0^{\pi} j_m(\theta) \sin\theta \, d\theta, \qquad j_m = \left.\frac{\partial n_m}{\partial \zeta}\right|_{\zeta=0} \tag{10}$$

$$u_1 \zeta^2 \frac{\partial n_m}{\partial \zeta} + u_2 \zeta \frac{\partial n_m}{\partial \theta} - \frac{\partial^2 n_m}{\partial \zeta^2} = F(\theta, \zeta) + \operatorname{ctg}\theta \frac{\partial n_{m-2}}{\partial \theta} \quad (m \geqslant 2)$$

In expansion (9), N is finite (we note however, that it is clearly no smaller than unity). Because of (8), there must exist some $m_0 > 1$ such that, when $m > m_0$, the quantity $j_m \sin\theta$ will have a nonintegrable singularity as $\theta \to \pi$.

For small R, we can use the Oseen approximation for u, which gives the term following the Stokes term in the asymptotic expansion in R. We find

$$u_2 \approx {}^3/_2 (1 + {}^3/_8 R) \sin\theta + {}^3/_4 R \sin\theta \cos\theta$$

$$\mu = \frac{3}{2} \frac{1 - {}^3/_4 R \cos\theta}{1 + {}^3/_8 R} \frac{2\theta - \sin 2\theta + {}^2/_3 R \sin^3\theta}{\sin^3\theta}$$

$$I \approx 7.848 (1 + {}^1/_8 R) \lambda^{-2/3} \tag{11}$$

According to [4], the term following the Oseen term is of the order of $R^2 \ln R$, so we have

$$u_2 = {}^3/_2 (1 + {}^3/_8 R + {}^9/_{40} R^2 \ln R) \sin\theta + {}^3/_4 R \sin\theta \cos\theta + O(R^2)$$

$$\mu \approx \frac{3}{2} \frac{1 - {}^3/_4 R \cos\theta}{1 + {}^3/_8 R + {}^9/_{40} R \ln R} \frac{2\theta - \sin 2\theta + {}^2/_3 R \sin^3\theta}{\sin^3\theta}$$

$$I \approx 7.848 (1 + {}^1/_8 R + {}^3/_{40} R^2 \ln R + O(R^2)) \lambda^{-2/3} \tag{12}$$

For moderate Reynolds numbers, we can use the numerical calculations of Jenson [5]. It is not difficult to show that $u_2 = \omega_0$, where ω_0 is the vorticity of velocity field **u** at the surface of the sphere. Values of ω_0 were given in [5] for R = 2.5, 5, and 10 at steps of 12° and for R = 20 at steps of 6°. Figure 1 shows dependences of the quantity $1/2 \sin\theta\, \omega_0$ on θ constructed from these data; curves 1-5 correspond to R = 0, 2.5, 5, 10, and 20, respectively. The θ dependence of j_0 is shown in Fig. 2; plots a-d correspond to R = 2.5, 5, 10, and 20, respectively. We note that in the calculation of j_0 for angles near π, a very large diffusive transport of the substance was assumed across the boundary of the steady-state eddy region. This assumption significantly increases j_0 for these angles, for the diffusive transport is actually small, while there is no convective transport into the eddy region, since at these R there is a steady-state flow around an object of complicated shape – the sphere and the connected eddy ring, without a flow of liquid into the leading part of the eddy ring or into the "bottom" region. Convective transport into the wake region will occur at larger R, when the flow becomes markedly nonsteady-state. However, despite this fictitious increase in j_0, Fig. 2 shows that its value at the "bottom" of the sphere is negligibly small in comparison with that at the leading part of the sphere.

Numerical integration of j_0 over the sphere yields the integral flow for R = 2.5, 5, 10, and 20. The results are approximated well by the analytic expression

$$I \approx 7.848 \ (1 + 0.127 \ R^{1/2}) \lambda^{-2/3} + O(\lambda^{-1}) \tag{13}$$

from which it follows that the relative increase in I as R changes from 0 to 20 is ≈ 30%. It should be noted that this change in I is small for such a large change in R.

Deformation of the spherical surface has an interesting effect on the diffusive flow. We consider deposition on an ellipsoid of revolution at R → 0 (the Stokes approximation). For the I calculations for b (the ratio of the longitudinal axis to the transverse axis) not too small, it is convenient to use the coordinate system

$$x = x_1^{(0)} - \xi\cos\theta\cos\varphi, \quad y = x_2^{(0)} + \xi\sin\theta\cos\varphi$$
$$z = x_3^{(0)} + \xi\sin\theta\sin\varphi, \quad 0 \leqslant \xi < \infty, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi$$
$$x_1^{(0)} = -\frac{b^2}{\sqrt{b^2 + \mathrm{tg}^2\theta}}, \quad \frac{x_2^{(0)}}{\cos\varphi} = \frac{x_3^{(0)}}{\sin\varphi} = \frac{\mathrm{tg}\,\theta}{\sqrt{b^2 + \mathrm{tg}^2\theta}}$$
$$L_\xi = 1, \quad L_\theta = \frac{b^2}{\cos^2\theta\,(b^2 + \mathrm{tg}^2\theta)^{3/2}} + \xi, \quad L_\varphi = \frac{\mathrm{tg}\,\theta}{\sqrt{b^2 + \mathrm{tg}^2\theta}} + \xi\sin\theta$$

Here L_ξ, L_θ, L_φ, are the Lamé coefficients. In this coordinate system, as in the case of the sphere, we can find an equation for the diffusion boundary layer. The solution of this equation for b not too small is again of the form (6), but with a different μ value. In this case, we have

$$\mu = 3b^3\frac{(1+\alpha)^3}{\sin^3\theta}\left\{\mathrm{arc\,tg}\left(\frac{\mathrm{tg}\,\theta}{b}\right) - b\,\frac{\mathrm{tg}\,\theta}{b^2 + \mathrm{tg}^2\theta}\right\}$$
$$(1+\alpha)^3 = \frac{4}{3}(1-b^2)\left\{1 + \frac{1-2b^2}{b\sqrt{1-b^2}}\,\mathrm{arc\,ctg}\,\frac{b}{\sqrt{1-b^2}}\right\}^{-1}$$
$$j_0 = \frac{1}{\Gamma(1+1/3)}\mu^{-1/3}$$
$$I = 2\pi\lambda^{-2/3}\left[\int_0^\pi d\theta\,\frac{(L_\theta L_\varphi)_{\xi=0}}{\mu^{1/3}}\right]^{2/3} + O(\lambda^{-1}) = 7.848\,(1+\alpha)\,\lambda^{-2/3} + O(\lambda^{-1}) \tag{14}$$

Figure 3 shows the dependence of a on b. We note that as b increases there is a considerable deformation of the thickness of the deposition at the leading part of the sphere: the thickness increases in the lateral regions and decreases at the center (Fig. 4 shows the dependence of j_0 on θ, illustrating this effect; the thickness is given for a diffusive deposit on an ellipsoid of revolution for b = 0.4).

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